MATH 1010E Lecture Notes Week 10 (Martin Li)

Last time ... Definite integral as Riemann sum.

$f : [a,b] \to \mathbb{R}$ continuous

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$$

Q: Any other efficient ways to calculate $\int_a^b f(x)\,dx$?

A: Yes. Fundamental Theorem of Calculus!

Fund. Thm. of Calculus I

If $F : [a,b] \to \mathbb{R}$ cts on $[a,b]$ & diff. in (a,b).

and $F'(x) = f(x)$ on (a,b).

then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} \quad \text{---} (*)$$

Notes: (1) Integration constant cancels out on the R.H.S.

e.g. $(x)' = 1$ and $(x+1)' = 1$

$$\Rightarrow \int_0^1 1\,dx = [x]_0^1 = 1 - 0 = 1$$

$$= [x+1]_0^1 = 2 - 1 = 1$$

(2) Rewrite $(*)$ as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

1st differentiate then integrate = "original function"

Examples:

Q: Primitive function of $x = \int x\,dx = \frac{x^2}{2} + C = F(x)$

(1) $\frac{1}{2} = \int_0^1 x\,dx \overset{(*)}{=} F(1) - F(0) = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$ #

take $F(x) = \frac{x^2}{2}$

(2) $\int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1} - \frac{0}{n+1} = \frac{1}{n+1}$ #.

$(n \neq -1)$

Ex: If we use Riemann sum, then we need a formula

for $1^n + 2^n + 3^n + \cdots + k^n = ?$

More Examples

(1) $\int_1^3 \underbrace{\frac{3x^3 - 5}{x^2}}_{\text{cts on } [1,3]}\,dx = \int_1^3 \left(3x - \frac{5}{x^2}\right)dx$

$= \left[\frac{3x^2}{2} + \frac{5}{x}\right]_1^3 = \left(\frac{27}{2} + \frac{5}{3}\right) - \left(\frac{3}{2} + 5\right)$

$= 7 + \frac{5}{3} = \frac{26}{3}$ #.

(2) $\int_0^1 x\sqrt{1-x^2}\,dx = \int_1^0 -\frac{1}{2}\sqrt{u}\,du = \int_0^1 \frac{1}{2}\sqrt{u}\,du$

u-substitution!
$\begin{cases} u = 1 - x^2 \\ du = -2x\,dx \end{cases}$

$x = 0 \leftrightarrow u = 1$
$x = 1 \leftrightarrow u = 0$

$= \left[\frac{1}{2}\,\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{1}{3}$ #.

(3) $$\int_0^{\pi/4} \underbrace{\sec^2 x \tan x}_{\text{cts on } [0, \frac{\pi}{4}].} \, dx = \int_0^{\pi/4} \tan x \, d(\tan x)$$

$$= \left[\frac{1}{2} \tan^2 x \right]_{0=x}^{\pi/4 = x}$$

$$= \frac{1}{2} - 0 = \frac{1}{2} \quad \divideontimes.$$

$$\int_0^{\pi/4} \sec^2 x \tan x \, dx = \int_0^{\pi/4} \sec x \, (\sec x \tan x) \, dx$$

$$= \int_0^{\pi/4} \sec x \, d(\sec x).$$

$$= \left[\frac{1}{2} \sec^2 x \right]_0^{\pi/4} = \frac{1}{2} \cdot 2 - \frac{1}{2} \cdot 1 = \frac{1}{2} \quad \divideontimes$$

Topics for Quiz 2

- L'Hospital's Rule : $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$ $\left(\frac{0}{0}, \frac{\infty}{\infty}, 0^0, 0 \cdot \infty \right)$

- implicit differentiation : $f(x, y) = 0$

  find y' in terms of x & y.

  at $(1, 0)$

- mean value theorems.

  $a < b$, $\quad \frac{f(b) - f(a)}{b - a} = f'(\xi)$ for some $\xi \in (a, b)$.

  $$\Rightarrow \boxed{f(b) = f(a) + f'(\xi)(b - a)}$$

- optimization : $\min/\max\limits_{x \in I} f(x)$

  $$\left.\begin{array}{l} \text{- } 1^{st} \text{ order condition : } f'(x_0) = 0 \text{ in interior} \\ \text{- boundary pts} \end{array}\right\} \text{compare.}$$

  - 2^{nd} derivative test : $f''(x_0) > 0 \Rightarrow \underset{+\,+}{\smile} \Rightarrow$ local min.

    $f''(x_0) < 0 \Rightarrow \overset{-\,-}{\frown} \Rightarrow$ local max.

- know before Quiz 1.

  e^x, $\sin x$, $\cos x$, ...

2. (g) Calculate y', y'' at $(1, \sqrt{3})$ for y defined implicitly by

$$x^2+y^2=4.$$

diff. once,

$$2x+2yy'=0$$

$$\Rightarrow \quad \boxed{y'=-\frac{x}{y}} \qquad \text{at } (1,\sqrt{3}),\ y'=-\frac{1}{\sqrt{3}}$$

diff again

$$y''=-\frac{y-xy'}{y^2}$$ ← involves x, y, y'.

$$\text{at } (1,\sqrt{3}), \quad =-\frac{\sqrt{3}-(1)(-\frac{1}{\sqrt{3}})}{3}$$

5. (a) $$f(x) = \frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}.$$

$1 \qquad\qquad x f(x)$

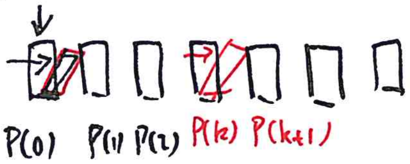
quotient rule: $$f'(x) = \frac{\sqrt{1+x^2}\cdot \dfrac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} - \ln(x+\sqrt{1+x^2})\dfrac{x}{\sqrt{1+x^2}}}{1+x^2}$$

$$\Rightarrow \boxed{(1+x^2)f'(x) + x f(x) = 1} \quad \text{—— } (*)$$

#

(b) Mathematical Induction: statement $P(n)$.

$$\begin{cases} n=0 & \text{true for } P(0) \quad \checkmark \\ n=k \text{ true} & \Rightarrow n=k+1 \text{ true.} \end{cases}$$



$\Rightarrow$ $P(n)$ true for all integer $n \geq 0$.

Now, back to this question,

$n=0$: " $\boxed{(1+x^2)f''(x) + 3xf'(x) + f(x) = 0}$."

diff. $(*)$. $[(1+x^2)f''(x) + 2xf'(x)] + [xf'(x) + f(x)] = 0$

$\Rightarrow$ holds for $n=0$.

$n=k \Rightarrow n=k+1$: Assume $n=k$ true.

ie $$(1+x^2)f^{(k+2)}(x) + (2k+3)xf^{(k+1)}(x) + (k+1)^2 f^{(k)}(x) = 0$$

diff. implicitly,

$$\Rightarrow (1+x^2)f^{(k+3)}(x) + 2xf^{(k+2)}(x) + (2k+3)xf^{(k+2)}(x)$$
$$+ (2k+3)f^{(k+1)}(x) + (k+1)^2 f^{(k+1)}(x) = 0$$

$$\Rightarrow (1+x^2)f^{(k+3)}(x) + (2(k+1)+3)xf^{(k+2)}(x)$$
$$+ [\underbrace{(k+1)^2+(2k+3)}_{k^2+4k+4 \,=\, (k+2)^2}]f^{(k+1)}(x) = 0$$

$\Rightarrow$ $n=k+1$ true.

By M.I. $\Rightarrow$ done!

11. (j). $\lim\limits_{x \to 0} \dfrac{x \tan x}{1-\sqrt{1-x^2}} = \lim\limits_{x \to 0} \dfrac{\tan x + x \sec^2 x}{\dfrac{x}{\sqrt{1-x^2}}}.$

$\dfrac{0}{0}$

$= \lim\limits_{x \to 0} \sqrt{1-x^2} \left(\dfrac{\tan x + x \sec^2 x}{x} \right)$

$= 1 \cdot \left(\lim\limits_{x \to 0} \dfrac{\tan x}{x} + 1 \right)$

$= \lim\limits_{x \to 0} \dfrac{\sec^2 x}{1} + 1$

$= 1 + 1 = 2$ ※.

15. (a) $\lim\limits_{x \to +\infty} \dfrac{x + \sin x}{x - \sin x} \neq \underbrace{\lim\limits_{x \to \infty} \dfrac{1+\cos x}{1-\cos x}}_{\text{limit does not exists!}}$

$\left(\dfrac{+\infty}{+\infty}\right)$

$= \lim\limits_{x \to +\infty} \dfrac{1 + \dfrac{\sin x}{x}}{1 - \dfrac{\sin x}{x}} = \dfrac{1+0}{1-0} = 1$ ※.

$\lim\limits_{x \to \infty} \dfrac{\sin x}{x} \overset{?}{=} 0$ by sandwich theorem

$\left| \dfrac{\sin x}{x} \right| \leq \dfrac{1}{|x|} \xrightarrow{x \to +\infty} 0$